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The Hermitian ovoids of Cossidente, Ebert, Marino, Siciliano

Norman L. Johnson

Department of Mathematics, University of Iowa,
Iowa City, Iowa 52242, USA
njohnson@math.uiowa.edu

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Abstract. The two classes of semifield spreads of Cossidente et al. arising from their Hermitian ovoids are completely determined as a set of Kantor-Knuth flock semifield spreads and a set of Hughes-Kleinfeld semifield spreads.

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1 Introduction

A Hermitian surface $\mathcal{H}$ in $PG(3, q^2)$ is the set of all isotropic points of a unitary polarity (non-degenerate), and an ovoid of $\mathcal{H}$ is a set of $q^3 + 1$ points which has exactly one common point with each of the generators (lines meeting $\mathcal{H}$ in $q^2 + 1$ points). An ovoid is a 'translation ovoid' with respect to one of its points P if and only if there is a group of order q^3 that fixes P, fixes each generator on P and acts transitively on the remaining points. By work of Thas [12] and generalized by Lunardon [10], there are close connections between semifield planes with spreads in $PG(3, q)$ and translation ovoids of a Hermitian surface, one giving rise to another. Recently, Cossidente, Ebert, Marino and Siciliano [4] have determined two new classes of such translation ovoids, each producing and equivalent to a class of semifield spreads in $PG(3, q)$. Cossidente, Ebert, Marino and Siciliano [4] show that one of these classes is a conical flock of semifield type (corresponds to a flock of a quadratic cone) and naturally is of great interest.

In this article, it is shown that the semifield flock spread found by Cossidente, Ebert, Marino and Siciliano corresponds to a class of Kantor-Knuth semifield flock spreads and the remaining class is a subclass of the semifields of Hughes-Kleinfeld.

2 Background

The connections with semifield spreads and translation ovoids of the Hermitian surface are intertwined with 'Shult' sets, which are, in fact, the duals of indicator sets. So, we provide a little background on indicator sets. Also, indicator sets are connected with transversal extensions of derivable nets. In this extension theorem, a dual translation plane is constructed from a transversal to a derivable net. The dual of this plane is a translation plane whose 'transpose' is the translation plane arising directly from an indicator set. This idea is key in understanding how to interpret the ovoids of Cossidente, Ebert, Marino and Siciliano.

All of this background material is taken from the Handbook of Finite Translation Planes by Biliotti, Jha and Johnson, which will appear in 2006, wherein the reader will find the details and proofs of this material.

2.1 Indicator sets

Indicator sets provide an alternative manner of determining spreads and were developed initially by R.H. Bruck. We begin with indicator sets which produce spreads of $PG(3,q)$.

Consider a 4-dimensional vector space V over a field K isomorphic to $GF(q)$. Form the tensor product of V with respect to a quadratic field extension F of K, F isomorphic to $GF(q^2)$, $V \otimes_K F$. If we form the corresponding lattices of subspaces to construct $PG(3,F)$, we will have a $PG(3,K)$ contained in $PG(3,F)$ such that, with respect to some basis for V over K, (x_1,x_2,y_1,y_2) for $x_1,x_2,y_1,y_2 \in K$ represents a point homogeneously in both $PG(3,K)$ and $PG(3,F)$. The Frobenius automorphism mapping defined by

$$\rho_q : (x_1,x_2,y_1,y_2) \longmapsto (x_1^q,x_2^q,y_1^q,y_2^q)$$

is a semi-linear collineation of $PG(3,F)$ with set of fixed points exactly $PG(3,K)$. We use the notation $Z^q = (x_1,x_2,y_1,y_2)^q = (x_1^q,x_2^q,y_1^q,y_2^q)$. Finally, choose a line $PG(1,K)$ within any given $PG(2,K)$ in the analogous manner so that there is a corresponding $PG(1,F)$. Hence, we have

$$\begin{aligned} &PG(1,K) \subseteq PG(3,K), \\ &PG(1,K) \subseteq PG(1,F) \subseteq PG(2,F). \end{aligned}$$

Now choose a $PG(2,F)$ such that $PG(2,F) \cap PG(3,K) = PG(1,K)$. For example, take $PG(2,F)$ as the lattice arising from the 3-dimensional vector space

$$\langle (1,0,0,0),(0,1,0,0),(0,0,1,e) \rangle \,;\; e \in F-K.$$

Then we note that that given any point Z of $\mathrm{PG}(2,F)-\mathrm{PG}(1,F)$, the space $\langle Z, Z^q\rangle$ is a 2-dimensional F-vector subspace, since if $Z^q=\lambda Z$, for $\lambda\in F$, then projectively $Z=Z^q$ and $Z\in\mathrm{PG}(3,K)$. Note if $e^q=e\alpha_0+\beta_0$, for $\alpha_0\neq 0$ and $\beta_0\in K$, then $(0,0,1,e)^q=(0,0,1,e\alpha_0+\beta_0)$. Since $e^q+e\in K$, it follows that $\alpha_0=-1$. Now we have $\langle(0,0,1,e),(0,0,1,e)^q=(0,0,1,-e+\beta_0)\rangle$. Hence, within this subspace is $(0,0,2,\beta_0)$. For example, if q is odd, we may take $\beta_0=0$, implying that $(0,0,1,0)$ is in the subspace. But this then implies that $(0,0,0,e)$ and hence $(0,0,0,1)$ is in the subspace. This means that $\langle(0,0,1,e),(0,0,1,e)^q\rangle\cap V/K=\langle(0,0,1,0),(0,0,0,1)\rangle$ is a 2-dimensional K-subspace which has trivial intersection with $\langle(1,0,0,0),(0,1,0,0)\rangle$.

More generally, $\langle Z,Z^q\rangle\cap V/K$ is a 2-dimensional K-vector subspace for all $Z\in\mathrm{PG}(2,F)-\mathrm{PG}(1,F)$.

1 Definition. A space $\mathrm{PG}(2,F)$ with the above properties (i.e., contains $\mathrm{PG}(1,K)\subseteq\mathrm{PG}(1,F)$, $\mathrm{PG}(2,F)\cap\mathrm{PG}(3,K)=\mathrm{PG}(1,K)$ so $\langle Z,Z^q\rangle\cap\mathrm{PG}(3,K)$ is a line skew to $\mathrm{PG}(1,F)$, for $Z\in\mathrm{PG}(2,F)-\mathrm{PG}(1,F)$) is called an 'indicator space'.

2 Definition. Let $\mathrm{PG}(2,F)$ be an indicator space within $\mathrm{PG}(3,F)$. An 'indicator set' I of $\mathrm{PG}(2,F)$ is a set of q^2 points in $\mathrm{PG}(2,F)-\mathrm{PG}(1,F)$ such that the line AB, for all $A\neq B\in S$, intersects $\mathrm{PG}(1,F)-\mathrm{PG}(1,K)$. Note that $\langle A,A^q\rangle$ now becomes a line of $\mathrm{PG}(2,F)$, which intersects $\mathrm{PG}(3,K)$ in a line skew to $\mathrm{PG}(1,F)$.

If $\mathrm{PG}(2,F)$ is an indicator space then $\mathrm{PG}(2,F)^q$ is an indicator space such that $\mathrm{PG}(2,F)^q\cap\mathrm{PG}(2,F)=\mathrm{PG}(1,F)$. Hence, if I is an indicator set, then, for $A\neq B$ of I, $\langle A,B\rangle\cap\langle A,B\rangle^q$ is a point of $\mathrm{PG}(1,F)-\mathrm{PG}(1,K)$.

We then arise at the following fundamental theorem.

3 Theorem. *If I is an indicator set then*

$$\{\,\langle A,A^q\rangle\cap\mathrm{PG}(3,K); A\in I\,\}\cup\mathrm{PG}(1,K)$$

is a spread of $\mathrm{PG}(2,K)$.

2.2 Transversals to derivable nets

Now we consider the affine version of the above result with a different visual image. Consider a finite derivable net. By the work of Johnson and De Clerck [5] and of Johnson [8], every finite derivable net is a regulus net. Hence, there is an ambient 4-dimensional vector space V over a field K isomorphic to $\mathrm{GF}(q)$ such that the derivable net, when affinely presented, corresponds to $\mathrm{PG}(1,q)$ on a Desarguesian line $\mathrm{PG}(1,q^2)$ and may be given the following partial spread representation:

$$N: x=0,\ y=x\alpha;\ \alpha\in K\simeq\mathrm{GF}(q).$$

We may thus consider the Desarguesian affine plane of order q^2, coordinatized by a quadratic field extension $F \supseteq K$ and isomorphic to $\mathrm{GF}(q^2)$. Now take an indicator set I of q^2 'affine' points. Without loss of generality, take I to contain the zero vector of the associated vector space. Take any line of the derivable net (regulus net) N, either $y = x\alpha + b$, for $\alpha \in K$ and $b \in F$, or $x = c$, for $c \in F$. Consider the points A_i, $i = 1, 2, \ldots, q^2$ of I, fix a point A_1, and consider the Desarguesian lines A_1A_i, $i = 2, 3, \ldots, A_{q^2}$. We know that these Desarguesian lines do not lie on a parallel class of N and each line must intersect each line of a parallel class of N in a unique point. Let λ be a parallel class of N and let ℓ be a line of λ. There are q^2 lines of λ each of which can contain at most one line of I, since I is an indicator set. However, on a given Desarguesian line A_1A_i, there are at least two lines of λ, say ℓ_1 and ℓ_i, each of which shares exactly one point of I. Now consider A_1A_j, for $i \neq j$, and let ℓ_j share A_j. If $\ell_i = \ell_j$, we have the obvious contradiction. Hence, there are $q^2 - 1 + 1$ different lines of λ that intersect I in exactly one point. Recall the following definition:

4 Definition. A 'transversal' T to a net $\mathcal{N}$ is a set of net points with the property that each line of the net intersects T in a unique point and each point of T lies on a line of each parallel class of $\mathcal{N}$.

Clearly, an indicator set provides a transversal to a derivable net and hence produces a dual translation plane π whose dual translation plane has a spread in $\mathrm{PG}(3, K)$. We now consider the converse. Assume that T is a transversal to a finite derivable net. Think of the scenario depicted previously, so we may consider T to be a set of q^2 points of a Desarguesian affine plane coordinatized by a field $F \supseteq K$, where K is a field isomorphic to $\mathrm{GF}(q)$, which coordinatizes the derivable net as a regulus net. We have an natural associated 4-dimensional K-vector space V in which the derivable net N lives and a natural 3-dimensional projective space $\mathrm{PG}(3, K)$. Form $V \otimes_K F$ and construct the natural 3-dimensional projective space $\mathrm{PG}(3, F)$ containing $\mathrm{PG}(3, K)$, think of the Desarguesian affine plane projectively as a $\mathrm{PG}(2, K)$, and embed this within $\mathrm{PG}(2, F)$ contained in $\mathrm{PG}(3, F)$ so that $\mathrm{PG}(2, F) - \mathrm{PG}(1, F)$ (the original line at infinity) contains no points of $\mathrm{PG}(3, K)$. Consider two distinct points A and B of T. Suppose that the Desarguesian line AB is in a parallel class of the derivable net N. Then there is a line of the net N which contains two points of the transversal, a contradiction. Hence, AB intersects 'outside' and hence in $\mathrm{PG}(1, F) - \mathrm{PG}(1, K)$. That is, a transversal to a derivable net produces an indicator set. Hence, we have proven the following theorem.

5 Theorem (also see Bruen [3])**.** *Finite indicator sets of* $\mathrm{PG}(2, q^2)$ *are equivalent to transversals to derivable nets.*

From the main section in transversal extension theory, we know that there is a spread arising geometrically from the embedding of the derivable net combinatorially in $\mathrm{PG}(3, K)$. We know that the corresponding translation plane is dual to the one which we obtain considering the extension theory using a transversal. However, if we use the transversal as an indicator set, there is another spread of $\mathrm{PG}(3, K)$. The question is: Is this spread dual to the associated dual translation plane? Normally the answer to this question is no! What actually occurs is that the 'transversal spread', the spread arising from the transversal extension theory, is 'dual' to the spread arising from the indicator spread, say the 'indicator spread'.

6 Theorem (Bruen [3])**.** *The indicator spread and transversal spreads are dual to each other.*

Now again consider a derivable net D and a transversal T to D. It has been pointed out in Johnson [8], that the way that the derived net D^* is related to the derivable net D using the geometric embedding in a 3-dimensional projective space $\mathrm{PG}(3, q)$ is that the structure is determined by a polarity of the original $\mathrm{PG}(3, q)$; to abuse the language, 'derivation is a polarity'. If we derive the net D to D^*, then T becomes a transversal to D^*, as can be seen by re-considering D^* in the standard form and embedding in a different $\mathrm{PG}(2, K^*)$. That is, T remains a set of q^2 points such that the new Desarguesian line AB, for $A, B \in T$, is not in the set of parallel classes defining the new derivable net N^*. To see this, we note that a line of D^* is a Baer subplane of D and in the original Desarguesian plane $AG(2, F)$, a Desarguesian line AB which is not in the parallel classes of D must intersect a Baer subplane in a unique point. Now the counting argument establishing that we have the appropriate intersection for an indicator set to be a transversal works here as well and shows that T is a transversal to D^*. Now consider the geometric spread (obtained by considering T as a set of lines of the new $\mathrm{PG}(3, K^*)$) so that $T \cup N^*$ (the adjoined line) becomes a spread of the dual space to $\mathrm{PG}(3, K)$.

Hence, we have the following connections:

7 Theorem. *Let D be a finite derivable net and let T be a transversal to D.*

(1) Form the transversal spread π_D^T (i.e., form the dual translation plane, dualize and find the spread within $\mathrm{PG}(3, K)$*, for K isomorphic to* $\mathrm{GF}(q)$*).*

(2) Realize T as an indicator set and form the indicator spread π_D^I.

(3) Derive D to D^* and, realizing that T is a transversal to D^*, form the transversal spread $\pi_{D^*}^I$.

(4) Realize T as an indicator set relative to D^* and form the indicator spread $\pi_{D^*}^I$.

(5) *Form from D and the transversal T the geometric extension spread π_D^{GT} obtained by realizing D combinatorially within* $\mathrm{PG}(3,K)$*.*

(6) *Form from D^* and the transversal T the geometric extension spread $\pi_{D^*}^{GT}$ obtained by realizing D^* combinatorially within* $\mathrm{PG}(3,K)$ *using a polarity of* $\mathrm{PG}(3,K)$*.*

Then the transversal spreads are isomorphic to the geometric spreads, respectively, and are dual to each other.

The indicator spreads are dual to the transversal spreads, respectively.

Hence, a given indicator spread is isomorphic to the 'derived' version of the original transversal spread (by 'derived' we mean derive the net and use the original transversal).

8 Corollary. *The dual translation plane obtained from an indicator set I of* $\mathrm{PG}(2,q^2)$*, using I as a transversal function to the natural derivable net (regulus net) defined using* $\mathrm{PG}(1,q)$*, is the dual plane to the translation plane constructed from the spread in* $\mathrm{PG}(3,q)$ *using I as in Theorem 3.*

2.3 Hermitian ovoids

In this subsection, we connect spreads in $\mathrm{PG}(3,q)$ with certain ovoids ('locally Hermitian') of the Hermitian surface $H(3,q^2)$. The connection with ovoids of $H(3,q^2)$ is work due to Shult [11] and Lunardon [10]. We review only the part of Hermitian varieties required for our constructions. The text of Hirschfeld and Thas [6] provides all of the details.

9 Definition. Let V_{2k} be a $2k$-dimensional vector space over a field L isomorphic to $\mathrm{GF}(q^2)$. A 'Hermitian form' is a mapping s with the following properties:

$$\begin{aligned} s\colon V_{2k}\oplus V_{2k} &\longmapsto L,\\ s(x+w,y+z) &= s(x,y)+s(w,y)+s(x,z)+s(w,z),\\ s(cx,dy) &= cd^q s(x,y) \text{ and } s(x,y)=s(y,x)^q,\\ s(x_0,V_{2k}) &= 0 \text{ implies } x_0=0 \text{ (i.e., a 'non-Degenerate Hermitian form').} \end{aligned}$$

Now assume that $k=2$.

Given a Hermitian form (non-degenerate), and given a vector subspace S, form S^δ as follows:

$$S^\delta = \{\, v\in V;\ s(v,S)=0\,\}.$$

Then the mapping

$$S\longmapsto S^\sigma$$

is a polarity of the associated projective 3-space $\mathrm{PG}(3,q^2)$, which is said to be a 'Hermitian polarity' or 'unitary polarity'.

The subgroup of $\Gamma L(4,q^2)$ which preserves the Hermitian form is called the 'unitary group'. This group is denoted by $\Gamma U(4,q^2)$. The associated group

$$\Gamma U\left(4,q^2\right)/Z\left(\Gamma U\left(4,q^2\right)\right)$$

is called the 'projective unitary group'.

10 Definition. A subspace S is said to be 'totally isotropic' if and only if

$$S \cap S^{\delta} = S.$$

In $\mathrm{PG}(3,q^2)$, the set of totally isotropic points and totally isotropic lines form the point-line geometry $H(3,q^2)$, the 'Hermitian surface'.

Projectively there is a canonical form for $H(3,q^2)$:

$$\left\{ (x_1,x_2,x_3,x_4)\,;\; x_1x_4^q + x_2x_3^q + x_3x_2^q + x_4x_1^q = 0 \right\}.$$

An 'ovoid' in this setting is a set of q^3+1 points of $H(3,q^2)$ which forms a cover of the set of totally isotropic lines.

For any two points A and B of $H(3,q^2)$, the line AB contains $q+1$ or q^2+1 points of $H(3,q^2)$. A 'tangent line' to a point C of $H(3,q^2)$ is a line containing exactly one point of $H(3,q^2)$.

11 Definition. A 'tangent plane' to $H(3,q^2)$ at a point P of $H(3,q^2)$ is the image plane of a point under the associated polarity. This plane will intersect $H(3,q^2)$ in exactly $q+1$ lines of $H(3,q^2)$ incident with P.

12 Remark. Given a plane Π of $\mathrm{PG}(3,q^2)$, the plane Π intersects $H(3,q^2)$ either at a point, a line, or a unital, or is a tangent plane. Hence, if a plane intersects in a line at points not on that line, then the plane is a tangent plane.

By Johnson [7], we consider the combinatorial structure of 'P-points' as lines of N and 'P-lines' as points of D considered as the set of intersecting lines. Furthermore, we call the parallel classes of N the 'P-hyperplanes'. We embed this structurally in a projective 3-space $\mathrm{PG}(3,q)$ by adjoining a line N to all P-hyperplanes. In this way, a derivable net of parallel classes, lines, and points becomes the set of hyperplanes of $\mathrm{PG}(3,q)$ incident with a particular line N, points of $\mathrm{PG}(3,q)-N$, and lines of $\mathrm{PG}(3,q)$ which are skew to N. In this model, T becomes a set of lines of $\mathrm{PG}(3,q)$ whose union with N is a spread. We know that this spread is isomorphic to the transferal spread and is the dual spread of the associated indicator spread.

If we consider the corresponding $\mathrm{PG}(2,F)$ and consider T as an indicator set, form the dual plane again isomorphic to $\mathrm{PG}(2,F)$ so that T is now a set of

q^2 lines with the property that the join of any two distinct 'lines' A and B does not intersect the dual of $\mathrm{PG}(1,K)$. Note that we may consider the $\mathrm{PG}(1,K)$ as a 'Hermitian line'. This means that if we dualize $\Pi = \mathrm{PG}(2,F)$, we find that $\Pi \cap H(3,q^2) = \Delta$ is a set of $q+1$ lines and the dual of T, T^D, is a set of q^2 lines that do not contain the point Q, which is the $\mathrm{PG}(1,F)$, with the property that no two intersect on a line of Δ. Such a set of lines is said to be a 'Shult set' [11]. Shult sets are equivalent to certain ovoids of $H(3,q^2)$.

13 Definition. An ovoid Φ of $H(3,q^2)$ is a set of q^3+1 points that cover the set of totally isotropic lines of $\mathrm{PG}(3,q^2)$ (each totally isotropic line is incident with exactly one point of Φ). Note that there will be exactly $q+1$ totally isotropic lines incident with each point. Choose any two distinct points Q and Z of Φ; then there are $q+1$ points of $H(3,q^2)$ on the line QZ. If for a fixed point Q and for all $Z \in \Phi$, the points on QZ are in Φ, we call Φ a 'locally Hermitian' ovoid with respect to Q.

If a locally Hermitian ovoid with fixed point Q admits a group that leaves $H(3,q^2)$ invariant, fixes all lines of $H(3,q^2)$ incident with Q, and acts transitively on the remaining points of the ovoid, we say that the ovoid is a 'translation ovoid' (i.e., a 'locally Hermitian translation ovoid).

For the Shult set T^D arising from the indicator set T, take the set of polar lines $T^{D\delta}$, with respect to the Hermitian polarity δ. Shult [11] shows that $T^{D\delta}$ is a set of q^2 lines incident with Q, such that there are $q+1$ points of $H(3,q^2)$ on each such line and the union of this set of points of $H(3,q^2)$ forms a locally Hermitian ovoid. Conversely, any locally Hermitian ovoid of $H(3,q^2)$ forms a Shult set, which dualizes to an indicator set, which constructs an indicator spread. Furthermore, if the original indicator set is a vector space transversal, then the constructed locally Hermitian ovoid admits a collineation group which fixes all lines of Q and acts transitively on the remaining point of the ovoid: that is, the locally Hermitian ovoid becomes a translation ovoid.

Hence, the work of Shult [11] and Lunardon [10], together with our interpretation, produces the following connections.

14 Theorem. *(1) Locally Hermitian ovoids of $H(3,q^2)$ are equivalent to transversals of finite derivable nets; one constructs the other.*

(2) The associated indicator spread is a semifield spread if and only if the locally Hermitian ovoid is a translation ovoid.

3 The translation ovoids of Cossidente, Ebert, Marino and Siciliano

In the previous section, we have noted that translation ovoids of the Hermitian surface are equivalent to semifield spreads in $\mathrm{PG}(3,q)$, which are equivalent to what are called 'vector space transversals' to derivable nets, in the sense that such vector space transversals produce dual translation planes which are actually translation planes. If we dualize the coordinate structure of such dual translation planes and then transpose the corresponding matrix spread set, we arrive at the semifield planes arising from the indicator set and which then correspond directly to the translation ovoids of the Hermitian surface.

3.1 The flock semifield of Cossidente, Ebert, Marino and Siciliano

Let $\mathrm{PG}(3,q^2)$ have homogeneous coordinates (a,b,c), letting K be the corresponding field isomorphic to $\mathrm{GF}\left(q^2\right)$ and F the subfield isomorphic to $\mathrm{GF}(q)$. Assume that $q=p^{2e}$, for p odd, and $\mathrm{Tr}(x)$ is the trace function over the subfield P isomorphic to $\mathrm{GF}(p^e)$:

$$\mathrm{Tr}(x)=x+x^{p^2}+x^q+x^{qp^e},\ x\in K.$$

The way that Cossidente et al. consider this function is to find a Baer subline H of $\mathrm{PG}(3,q^2)$ which is disjoint from the set F_T, where

$$F_T=\{\,(1,a,\mathrm{Tr}(a));\ a\in K\,\}.$$

They find one as follows:

$$H=\{\,(0,0,1)\,\}\cup\{\,(0,1,z);\ z\in K \text{ and } z+z^q=4\,\}.$$

Now from the point of view considered previously, what they have found is a transversal to a derivable net. In order to apply our ideas, we need to recoordinatize the derivable net into what we call 'standard form' and then apply transversal extension theory. This will provide a dual translation plane. However, since the corresponding function is additive, we will actually construct a semifield plane. The semifield plane obtained by transposing the matrix spread set of this dual translation plane will be the spread constructed by Cossidente et al. We might point out that in Biliotti, Jha and Johnson [2], it is noted that any transposed flock spread is isomorphic to the original flock spread. Since what Cossidente et al. have found is a semifield flock spread, we need not transpose to find the corresponding spread.

We consider this from the affine point of view, which means we consider $\mathrm{PG}(3, q^2)$ as the ambient set of points of a 4-dimensional vector space over F with an ideal line adjoined. In other words, our points are the points of $AG(3, q^2)$, whose points we take as $(x, y) = (1, x, y)$ in the notation of Cossidente et al.

Hence, we have:

15 Lemma. *Consider the derivable net D*

$$D = \{ x = 0, y = xm;\ m^q + m = 4;\ m \in K \}.$$

Then the set

$$y = \mathrm{Tr}(x)$$

is a transversal to D.

In order to apply our methods, we transform D to the standard representation in two steps. First we note that $m = 2$ satisfies the condition. Let

$$\sigma : (x, y) \to (x, -2x + y)$$

be a coordinate change to bring $y = 2x$ to $y = 0$ while preserving $x = 0$. This transforms $y = \mathrm{Tr}(x)$ into $y = \mathrm{Tr}(x) - 2x$, while mapping D into

$$D\sigma = \{ x = 0, y = 0, y = x(m - 2);\ m^q + m = 4 \}.$$

Note that

$$m^q + m = 4 \iff (m - 2)^q + (m - 2) = 0.$$

Hence, we have proven the following:

16 Lemma. *We may represent the previous transversal function and derivable net, respectively, as*

$$\begin{aligned} y &= \mathrm{Tr}(x) - 2x, \\ D\sigma &= \{ x = 0, y = 0, y = xn;\ n^q + n = 0 \}. \end{aligned}$$

We now choose a basis $\{ e, 1 \}$ for K such that $e^2 = e\gamma$, where γ is a nonsquare in F. Note that it follows easily that $e^q + e = 0$. Hence, we have $y = xe$ in our derivable net in the current representation. Choose

$$\tau : (x, y) \to \left(x, ye^{-1}\right)$$

and note that $e^{-1} = e/\gamma = e\rho$, for $\rho = 1/\gamma$. Noting that once a derivable net (considered as a regulus net in the ambient Desarguesian affine plane) contains $x = 0, y = 0, y = x$, it does have the standard representation.

Hence, we have:

17 Lemma. *Under the basis change τ, we may represent the transversal function and derivable net, respectively, as*

$$y = (\mathrm{Tr}(x) - 2x)e\rho,$$
$$D\sigma\tau = \{\, x = 0, y = x\alpha;\ \alpha \in F \,\}.$$

If we have a transversal function $y = f(x)$ to a derivable net represented in the standard form then the following set of lines defines a dual translation plane:

$$\{\, x = 0, y = x\alpha + f(x)\beta + b;\ \alpha, \beta \in F,\ b \in K \,\}.$$

Hence, we obtain the following dual translation plane:

$$\{\, x = 0, y = x\alpha + (\mathrm{Tr}(x) - 2x)e\rho\beta + b;\ \alpha, \beta \in F,\ b \in K \,\}.$$

Now we note that this dual translation plane actually is a translation plane since the function $(\mathrm{Tr}(x) - 2x)e\rho$ is additive. However, this translation plane does not yet correspond to the semifield plane found by Cossidente et al., for we need to dualize and then transpose.

We do the dualization algebraically in the associated dual translation plane by taking a multiplication "$\circ$" as follows:

$$x \circ (e(-2\rho\beta) + \alpha) = x(e(-2\rho\beta) + \alpha) + \mathrm{Tr}(x)e\rho\beta$$

for all $x \in K$, $\beta, \alpha \in F$. To obtain the desired semifield plane, we form the opposite multiplication "$*$" by

$$x \circ (e(-2\rho\beta) + \alpha) = (e(-2\rho\beta) + \alpha) * x.$$

Now we perform a notation change: Let

$$ex_1 + x_2 = e(-2\rho\beta) + \alpha,$$
$$x = et + u,$$

where $x_1, x_2, t, u \in F$. Hence, we have

$$(ex_1 + x_2) * (et + u) = (et + u)(ex_1 + x_2) + \mathrm{Tr}(et + u)e(-x_1/2)$$

(the ρ term drops out).

We now work out the trace function:

18 Lemma. $\mathrm{Tr}(et + u) = 2(u + u^{p^2})$.

Proof. $\mathrm{Tr}(et{+}u) = (et + u) + \big(e^{p^e}t^{p^e} + u^{p^e}\big) + (-et + u) + \big(-e^{p^e}t^{p^e} + u^{p^e}\big) = 2\,\big(u + u^{p^e}\big)$. $\boxed{QED}$

Hence, we obtain the following theorem.

19 Theorem. *The flock semifield of Cossidente, Ebert, Marino and Siciliano has the following spread set:*

$$\left\{ x = 0, y = x \begin{bmatrix} -u^{p^e} & \gamma t \\ t & u \end{bmatrix};\ u, t \in F \right\}.$$

PROOF.

$$\begin{aligned}(ex_1 + x_2) * (et + u) &= (et + u)(ex_1 + x_2) + \mathrm{Tr}(et + u)e(-x_1/2) \\ &= e\left(tx_2 - u^{p^e} x_1\right) + \gamma t x_1 + u x_2\end{aligned}$$

using $e^2 = e\gamma = e/\rho$. $\boxed{QED}$

Now we change bases again by

$$\omega : (x, y) \to \left(x, y \begin{bmatrix} 0 & 1 \\ \rho & 0 \end{bmatrix} \right)$$

and take $u\rho = v$, to transform the spread into

$$\left\{ x = 0, y = x \begin{bmatrix} t & -\gamma v^{p^e} \\ v & t \end{bmatrix};\ u, t \in F \right\}.$$

If we note that -1 is a square in $\mathrm{GF}(q)$ as q is a square, then $-\gamma = \gamma_0$ is a non-square. Hence, we finally transform the spread into

$$\left\{ x = 0, y = x \begin{bmatrix} t & \gamma_0 v p^e \\ v & t \end{bmatrix};\ u, t \in F \right\},$$

which is the standard representation of the Kantor-Knuth semifield flock spreads (see, e.g., Johnson and Payne [9]). Therefore, we have shown:

20 Theorem. *The flock semifield of Cossidente, Ebert, Marino and Siciliano is a Kantor-Knuth semifield flock spread.*

3.2 The second semifield of Cossidente, Ebert, Marino and Siciliano

In this case, the setup is exactly as in the previous subsection. Let F_B be defined as follows:

$$F_B = \left\{ \left(1, a, a^{p^f}\right);\ a \in K \right\},$$

for $q = p^{2e}$, q odd, and f a divisor of e. The associated Baer subline of $\mathrm{PG}(3, q^2)$ is U_B:

$$U_B = \{ (0, 0, 1)\} \cup \{(0, 1, z);\ z \in K;\ z + z^q = 0 \}.$$

Now we employ the ideas affinely and realize a derivable net as follows:

$$D = \{ x = 0, y = 0, y = xm;\ m^q + m = 0;\ m \in K - \{ 0 \} \} .$$

This time, we start with two components of the required standard representation. We again choose a basis $\{ e, 1 \}$ so that $e^2 = e\gamma$, for γ a non-square in F. Since in this case $e^q + e = 0$, we choose a new basis by the mapping

$$\delta : (x, y) \to (x, ye\rho),$$

where $\rho = 1/\gamma$. We then have the proof to the following lemma.

21 Lemma. *The following is a transversal to a derivable net in standard representation:*

$$y = x^{p^f} e\rho.$$

We again form the associated dual translation plane:

$$x = 0, y = x\alpha + x^{p^f} e\rho\beta + b;\ \alpha, \beta \in F,\ b \in K.$$

We choose a multiplication "$\circ$" for the dual translation plane and "$*$" for the dual of this plane (we are after the transposed version of this plane).

$$x \circ (e\rho\beta + \alpha) = x\alpha + x^{p^f} e\rho\beta = (e\rho\beta + \alpha) * x.$$

Now let

$$\begin{aligned} ex_1 + x_2 &= e\rho\beta + \alpha, \\ x &= et + u. \end{aligned}$$

Then, we obtain:

$$(ex_1 + x_2) * (et + u) = (et + u)x_2 + \left(e^{p^f} t^{p^r} + u^{p^f} \right) ex_1.$$

Let

$$e^{p^f} = eg + h,$$

to obtain:

$$\begin{aligned} (ex_1 + x_2) * (et + u) = (et + u)x_2 + \left((eg + h)\, t^{p^r} + u^{p^f} \right) ex_1 \\ e\left(tx_2 + \left(ht^{p^f} + u^{p^f} \right) x_1 \right) + \left(ux_2 + \gamma g t^{p^f} x_1 \right). \end{aligned}$$

Therefore, by transposing the resulting matrix spread set, we have the following theorem.

22 Theorem. *The second semifield plane of Cossidente, Ebert, Marino and Siciliano has spread set:*

$$\left\{ x = 0, y = x \begin{bmatrix} ht^{p^f} + u^{p^f} & t \\ \gamma g t^{p^f} & u \end{bmatrix};\ u, t \in F \right\}.$$

23 Corollary. *The second semifield plane of Cossidente, Ebert, Marino and Siciliano is the transposed dual of a Hughes-Kleinfeld semifield plane.*

To see this, from Biliotti, Jha, Johnson [1] we note the following:

In this setting, F is an arbitrary field isomorphic to $\mathrm{GF}(q)$ and θ is an automorphism of F, while $\{ \lambda, 1 \}$ is a basis for K over F.

24 Theorem (Hughes-Kleinfeld Semifelds)**.** *Suppose $a = x^{1+\theta} + xb$ has no solution for x in F. Then*

$$(x + \lambda y) \circ (z + \lambda t) = \left(xz + aty^{\theta} \right) + \lambda \left(yz + \left(x^{\theta} + y^{\theta} b \right) t \right)$$

is a semifield and F is its right and middle nucleus. Conversely, if D is a semifield that is a finite two-dimensional vector space over a field F such that the middle and right nucleus of D coincide, then D is a Hughes-Kleinfeld semifield.

Dualize the Hughes-Kleinfeld semifield in the manner above:

$$(z + \lambda t) * (x + \lambda y) = \left(xz + aty^{\theta} \right) + \lambda \left(yz + \left(x^{\theta} + y^{\theta} b \right) t \right).$$

Now let

$$y = s,\ x = u,$$

to transform the equation into

$$(z + \lambda t) * (u + \lambda s) = \left(uz + ats^{\theta} \right) + \lambda \left(sz + \left(u^{\theta} + s^{\theta} b \right) t \right)$$

and obtain the following matrix spread set:

$$\left\{ x = 0, y = x \begin{bmatrix} (u^{\theta} + s^{\theta} b) & as^{\theta} \\ s & u \end{bmatrix};\ u, s \in F \right\}.$$

Now transpose this matrix spread set to obtain

$$\left\{ x = 0, y = x \begin{bmatrix} (u^{\theta} + s^{\theta} b) & s \\ as^{\theta} & u \end{bmatrix};\ u, s \in F \right\},$$

and, comparing the semifield spread of Cossidente et al., we see that the spreads for $\theta = p^f$ are isomorphic. Then

$$\left\{ x = 0, y = x \begin{bmatrix} ht^{p^f} + u^{p^f} & t \\ \gamma g t^{p^f} & u \end{bmatrix};\ u, t \in F \right\}.$$

This completes the proof of the corollary.

25 Remark. Some of the results of this article have been proved independently by Cossidente, Lunardon, Marino, and Polverino, Hermitian indicator sets (to appear), by similar yet distinct methods. In particular, we have used the theory of transversals to derivable nets to prove our results, whereas Cossidente et al. use primarily analysis using indicator sets, which is a more general study. Their results identify the flock semifield planes as Kantor-Knuth using the connection that such flocks are precisely those whose planes (containing the conics) intersect in a point, whereas our study is algebraic. These works also generally overlap in that Cossidente et al. connect locally Hermitian ovoids with indicator sets, while here we phrase this using transversals to derivable nets.

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